Space Curves

Defⁿ: A <u>space curve</u> is a function taking an interval to vectors in n-space for some fixed n.

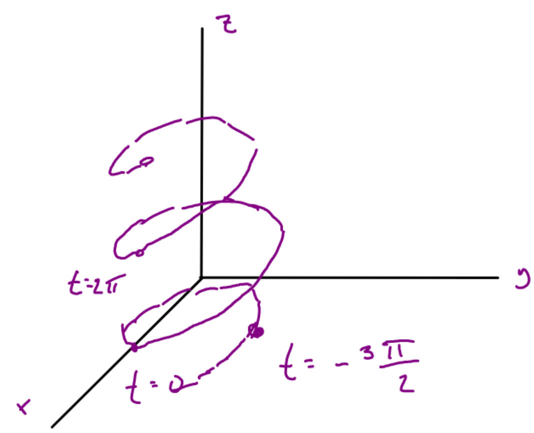
Notation: We usually write $\vec{r} : I \to \mathbb{R}^3$ to denote a space curve in 3-space.

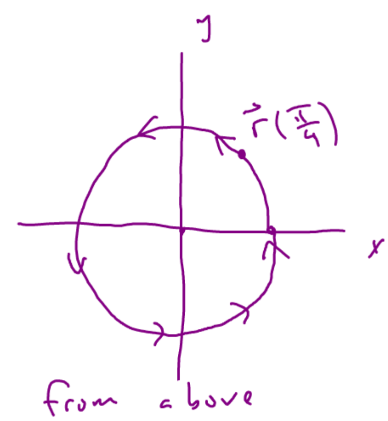
NB: Space curves are examples of <u>vector-valued functions</u>.

Fact: Every vector-valued function is determined by its component functions. Thus we can rewrite every space curve as $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ where $x, y, z : I \to \mathbb{R}$. This observation is key to working with space curves.

Ex: The curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ is the <u>helix</u>.

Sketch:





Ex: The curve below is the <u>trefoil knot</u> curve.

$$\vec{r}(t) = \left(\sin(t) + 2\sin(2t)\right)\vec{i} + \left(\cos(t) - 2\cos(2t)\right)\vec{j} + \sin(3t)\,\vec{k}$$

Ex: The curve below is the <u>moment curve</u>.

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle \quad \text{for } t \in [0, \infty).$$

Ex: The curve below is a toroidal curve (it winds around the surface of a torus).



$$\vec{r}(t) = \langle (4 + \sin(20t))\cos(t), (4 + \sin(20t))\sin(t), \cos(20t) \rangle$$

Ex: The <u>(p,q) - torus knot</u> is the curve below.

$$\vec{r}_{p,q}(t) = \langle \cos(qt)(3 + \cos(pt)), \sin(qt)(3 + \cos(pt)), \sin(pt) \rangle$$

Ex: Find a vector function representing the curve of intersection of the surfaces ...

① $x^2+y^2=4$ and $z=xy$

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

$$\begin{cases} x(t) = 2\cos(t) \\ y(t) = 2\sin(t) \\ z(t) = x(t)\,y(t) \end{cases}$$

(the first two ← parametrizes a circle of radius 2)

$\therefore \vec{r}(t) = \langle 2\cos(t), 2\sin(t), 4\cos(t)\sin(t)\rangle$ for $t \in [0, 2\pi]$.

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② $z=\sqrt{x^2+y^2}$ and $z=1+y$

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

$$\begin{cases} z=1+y \\ z=\sqrt{x^2+y^2} \end{cases} \iff \begin{cases} z=1+y \\ 1+y=\sqrt{x^2+y^2} \end{cases} \iff \begin{cases} x=t \\ y=\frac{1}{2}(t^2-1) \\ z=\frac{1}{2}(t^2+1) \end{cases}$$

$(1+y)^2 = x^2+y^2 \iff 1+2y+y^2 = x^2+y^2 \iff 1+2y=x^2$

$\therefore r(t) = \langle t, \frac{1}{2}(t^2-1), \frac{1}{2}(t^2+1)\rangle$

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Ex: At what points does the helix $\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$ intersect the sphere $x^2+y^2+z^2 = 5$?

Sol: At the time t of intersection,

$\vec{r}(t)$ satisfies $x(t)^2 + y(t)^2 + z(t)^2 = 5$

i.e. $\sin^2(t) + \cos^2(t) + t^2 = 5$

$\iff t^2 = 4 \iff t = \pm 2$

Hence $\vec{r}(2)$ and $\vec{r}(-2)$ are the points of intersection

i.e. $(\sin(2), \cos(2), 2)$ and $(\sin(-2), \cos(-2), -2)$ ☑

Ex: Particles travel in 3-space along the curves $\vec{\alpha}(t) = \langle t, t^2, t^3 \rangle$ and $\vec{\beta}(t) = \langle 1+2t, 1+6t, 1+14t \rangle$.
Do the particles collide? Do their paths intersect?

Sol: If their paths intersect, then $\vec{\alpha}(t) = \vec{\beta}(s)$ for some $s,t \in \mathbb{R}$.

$$\vec{\alpha}(t) = \vec{\beta}(s) \iff \begin{cases} t = 1+2s \\ t^2 = 1+6s \\ t^3 = 1+14s \end{cases} \iff \begin{cases} t = 1+2s \\ (1+2s)^2 = 1+6s \\ (1+2s)^3 = 1+14s \end{cases} \iff \begin{cases} t = 1+2s \\ 4s^2+4s+1 = 1+6s \\ (1+2s)^3 = 1+14s \end{cases}$$

$$\iff \begin{cases} t = 1+2s \\ 4s^2 - 2s = 0 \\ (1+2s)^3 = 1+14s \end{cases} \iff \begin{cases} t = 1+2s \\ 2s(2s-1) = 0 \\ (1+2s)^3 = 1+14s \end{cases} \iff \begin{cases} t = 1+2s \\ s = 0 \text{ OR } s = \frac{1}{2} \\ (1+2s)^3 = 1+14s \end{cases} \leftarrow$$

$$\iff \text{EITHER} \begin{cases} t = 1+0 \\ s = 0 \\ (1+0)^3 = 1+0 \end{cases} \text{ OR } \begin{cases} t = 1+2\cdot\frac{1}{2} \\ s = \frac{1}{2} \\ (1+2\cdot\frac{1}{2})^3 = 1+14\cdot\frac{1}{2} \end{cases}$$

$$\iff \text{Either} \begin{cases} t = 1 \\ s = 0 \end{cases} \text{ OR } \begin{cases} t = 2 \\ s = \frac{1}{2} \end{cases}$$

$\therefore$ The particle paths intersect at $\vec{\alpha}(1) = \vec{\beta}(0)$ and $\vec{\alpha}(2) = \vec{\beta}(\frac{1}{2})$, but they don't collide. ☒

Ex: Particles travel along paths given by the curves $\vec{\gamma}(t) = \langle t^2, 7t-12, t^2 \rangle$ and $\vec{\lambda}(t) = \langle 4t-3, t^2, 5t-6 \rangle$.
Do the particles collide? Do their paths intersect?

Ex: $\vec{\gamma}(s) = \vec{\lambda}(t) \iff \begin{cases} s^2 = 4t-3 \\ 7s-12 = t^2 \\ s^2 = 5t-6 \end{cases} \implies 4t-3 = s^2 = 5t-6 \implies 3 = t$

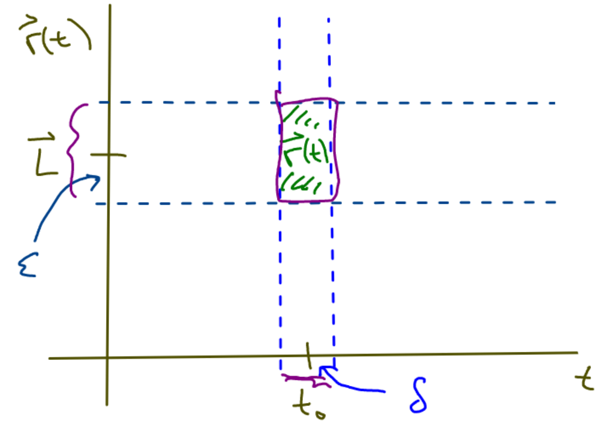
$$\implies \begin{cases} s^2 = 9 \\ 7s-12 = 9 \\ s^2 = 9 \end{cases} \iff \begin{cases} s = \pm 3 \\ 7s = 21 \end{cases} \iff \begin{cases} s = \pm 3 \\ s = 3 \end{cases} \iff s = 3$$

$\therefore \vec{\gamma}(3) = \vec{\lambda}(3)$ so the particles do collide ☒

Limits of Space Curves

Def

Defn: Let $\vec{r}: I \to \mathbb{R}^n$ be a space curve. The curve $\vec{r}(t)$ has limit $\vec{L}$ at $t = t_0$ when for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $t \in I$ we have $|t - t_0| < \delta$ implies $|\vec{r}(t) - \vec{L}| < \varepsilon$.



We write $\lim_{t \to t_0} \vec{r}(t) = \vec{L}$ or $\vec{r}(t) \to \vec{L}$ as $t \to t_0$.

NB: This is the formal definition of the limit of a space curve. For $n=1$ we recover limits from Calc 1.

Much like in Calculus 1, we will mostly ignore this definition. :(So how will we deal with limits?

Prop: The limit of a space curve is the vector with components the limits of its component functions, provided those limits exist.

I.E. If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then we have

$$\lim_{t \to t_0} \vec{r}(t) = \lim_{t \to t_0} \langle x(t), y(t), z(t) \rangle = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle.$$

Ex: Compute the indicated limits below (if they exist).

ⓐ $\lim_{t\to 0}\left(e^{-3t}\vec{i} + \frac{t}{\sin^2(t)}\vec{j} + \cos(2t)\vec{k}\right)$

$= \lim_{t\to 0}\left\langle e^{-3t}, \frac{t}{\sin^2(t)}, \cos(2t)\right\rangle$

$= \left\langle \lim_{t\to 0} e^{-3t}, \lim_{t\to 0}\frac{t}{\sin^2(t)}, \lim_{t\to 0}\cos(2t)\right\rangle$

$\overset{H}{=} \left\langle e^0, \lim_{t\to 0}\frac{1}{2\sin(t)\cos(t)}, \cos(0)\right\rangle$

B/C $\lim_{t\to 0}\frac{1}{2\sin(t)\cos(t)}$ DNE, this limit DNE!

L'H Rule: If $\frac{f}{g}\to\frac{0}{0}$ and "g is good" then $\lim\frac{f}{g} = \lim\frac{f'}{g'}$ ...

ⓑ $\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\vec{i} + \sqrt{t+8}\,\vec{j} + \frac{\sin(\pi t)}{\ln(t)}\vec{k}\right)$

$= \lim_{t\to 1}\left\langle \frac{t(t-1)}{t-1}, \sqrt{t+8}, \frac{\sin(\pi t)}{\ln(t)}\right\rangle$

$= \left\langle \lim_{t\to 1}\frac{t(t-1)}{t-1}, \lim_{t\to 1}\sqrt{t+8}, \lim_{t\to 1}\frac{\sin(\pi t)}{\ln(t)}\right\rangle$

$\overset{H}{=} \left\langle \lim_{t\to 1} t, \sqrt{1+8}, \lim_{t\to 1}\frac{\pi\cos(\pi t)}{1/t}\right\rangle$

$= \left\langle 1, \sqrt{9}, \frac{\pi\cos(\pi)}{1/1}\right\rangle = \langle 1, 3, -\pi\rangle$ ☒

ⓒ $\lim_{t\to\infty}\left(\frac{1+t^2}{1-t^2}\vec{i} + \arctan(t)\vec{j} + \frac{1-e^{-2t}}{t}\vec{k}\right)$

$= \lim_{t\to\infty}\left\langle \frac{1+t^2}{1-t^2}, \arctan(t), \frac{1-e^{-2t}}{t}\right\rangle$

$= \left\langle \lim_{t\to\infty}\frac{1+t^2}{1-t^2}, \lim_{t\to\infty}\arctan(t), \lim_{t\to\infty}\frac{1-e^{-2t}}{t}\right\rangle$

$= \left\langle \lim_{t\to\infty}\frac{\frac{1}{t^2}+1}{\frac{1}{t^2}-1}, \lim_{t\to\infty}\arctan(t), \lim_{t\to\infty}\frac{1-\frac{1}{e^{2t}}}{t}\right\rangle$

$= \left\langle \frac{0+1}{0-1}, \frac{\pi}{2}, 0\right\rangle = \left\langle -1, \frac{\pi}{2}, 0\right\rangle$ ☒

Prop (Properties of Limits): If scalar function $c : I \to \mathbb{R}$ and space curves $\vec{u}, \vec{v} : I \to \mathbb{R}^n$ have limits as $t \to t_0$, then:

① $\lim\limits_{t \to t_0} \left(\vec{u}(t) + \vec{v}(t) \right) = \lim\limits_{t \to t_0} \vec{u}(t) + \lim\limits_{t \to t_0} \vec{v}(t)$ ←

$= \lim\limits_{t \to t_0} \langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t) \rangle$

$= \langle \lim\limits_{t \to t_0} (u_1(t) + v_1(t)), \lim\limits_{t \to t_0} (u_2(t) + v_2(t)), \lim\limits_{t \to t_0} (u_3(t) + v_3(t)) \rangle$

$= \langle \lim\limits_{t \to t_0} u_1(t) + \lim\limits_{t \to t_0} v_1(t), \lim\limits_{t \to t_0} u_2(t) + \lim\limits_{t \to t_0} v_2(t), \lim\limits_{t \to t_0} u_3(t) + \lim\limits_{t \to t_0} v_3(t) \rangle$

$= \langle \lim\limits_{t \to t_0} u_1(t), \lim\limits_{t \to t_0} u_2(t), \lim\limits_{t \to t_0} u_3(t) \rangle + \langle \lim\limits_{t \to t_0} v_1(t), \lim\limits_{t \to t_0} v_2(t), \lim\limits_{t \to t_0} v_3(t) \rangle$

② $\lim\limits_{t \to t_0} c(t)\vec{v}(t) = \lim\limits_{t \to t_0} c(t) \lim\limits_{t \to t_0} \vec{v}(t)$ ←

$= \lim\limits_{t \to t_0} \langle c(t)v_1(t), c(t)v_2(t), c(t)v_3(t) \rangle$

$= \langle \lim\limits_{t \to t_0} c(t)v_1(t), \lim\limits_{t \to t_0} c(t)v_2(t), \lim\limits_{t \to t_0} c(t)v_3(t) \rangle$

$= \langle \lim\limits_{t \to t_0} c(t) \lim\limits_{t \to t_0} v_1(t), \lim\limits_{t \to t_0} c(t) \lim\limits_{t \to t_0} v_2(t), \lim\limits_{t \to t_0} c(t) \lim\limits_{t \to t_0} v_3(t) \rangle$

$= \lim\limits_{t \to t_0} c(t) \langle \lim\limits_{t \to t_0} v_1(t), \lim\limits_{t \to t_0} v_2(t), \lim\limits_{t \to t_0} v_3(t) \rangle$

③ $\lim\limits_{t \to t_0} \left(\vec{u}(t) \cdot \vec{v}(t) \right) = \lim\limits_{t \to t_0} \vec{u}(t) \cdot \lim\limits_{t \to t_0} \vec{v}(t)$ ←

$= \lim\limits_{t \to t_0} \left(u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t) \right)$

$= \lim\limits_{t \to t_0} u_1(t) \lim\limits_{t \to t_0} v_1(t) + \lim\limits_{t \to t_0} u_2(t) \lim\limits_{t \to t_0} v_2(t) + \lim\limits_{t \to t_0} u_3(t) \lim\limits_{t \to t_0} v_3(t)$

$= \langle \lim\limits_{t \to t_0} u_1(t), \lim\limits_{t \to t_0} u_2(t), \lim\limits_{t \to t_0} u_3(t) \rangle \cdot \langle \lim\limits_{t \to t_0} v_1(t), \lim\limits_{t \to t_0} v_2(t), \lim\limits_{t \to t_0} v_3(t) \rangle$

④ $\lim\limits_{t \to t_0} \left(\vec{u}(t) \times \vec{v}(t) \right) = \lim\limits_{t \to t_0} \vec{u}(t) \times \lim\limits_{t \to t_0} \vec{v}(t)$ ←

NB: $\lim\limits_{t \to t_0} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \lim\limits_{t \to t_0} (ad - bc) = \lim\limits_{t \to t_0} a \lim\limits_{t \to t_0} d - \lim\limits_{t \to t_0} b \lim\limits_{t \to t_0} c = \begin{vmatrix} \lim\limits_{t \to t_0} a & \lim\limits_{t \to t_0} b \\ \lim\limits_{t \to t_0} c & \lim\limits_{t \to t_0} d \end{vmatrix}$

$= \lim\limits_{t \to t_0} \left\langle \begin{vmatrix} u_2(t) & u_3(t) \\ v_2(t) & v_3(t) \end{vmatrix}, -\begin{vmatrix} u_1(t) & u_3(t) \\ v_1(t) & v_3(t) \end{vmatrix}, \begin{vmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{vmatrix} \right\rangle$

$= \left\langle \lim\limits_{t \to t_0} \begin{vmatrix} u_2(t) & u_3(t) \\ v_2(t) & v_3(t) \end{vmatrix}, -\lim\limits_{t \to t_0} \begin{vmatrix} u_1(t) & u_3(t) \\ v_1(t) & v_3(t) \end{vmatrix}, \lim\limits_{t \to t_0} \begin{vmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{vmatrix} \right\rangle$

$= \left\langle \begin{vmatrix} \lim\limits_{t \to t_0} u_2(t) & \lim\limits_{t \to t_0} u_3(t) \\ \lim\limits_{t \to t_0} v_2(t) & \lim\limits_{t \to t_0} v_3(t) \end{vmatrix}, -\begin{vmatrix} \lim\limits_{t \to t_0} u_1(t) & \lim\limits_{t \to t_0} u_3(t) \\ \lim\limits_{t \to t_0} v_1(t) & \lim\limits_{t \to t_0} v_3(t) \end{vmatrix}, \begin{vmatrix} \lim\limits_{t \to t_0} u_1(t) & \lim\limits_{t \to t_0} u_2(t) \\ \lim\limits_{t \to t_0} v_1(t) & \lim\limits_{t \to t_0} v_2(t) \end{vmatrix} \right\rangle$

$= \langle \lim\limits_{t \to t_0} u_1(t), \lim\limits_{t \to t_0} u_2(t), \lim\limits_{t \to t_0} u_3(t) \rangle \times \langle \lim\limits_{t \to t_0} v_1(t), \lim\limits_{t \to t_0} v_2(t), \lim\limits_{t \to t_0} v_3(t) \rangle$

CONTINUITY OF SPACE CURVES

Def^n: A space curve $\vec{r}(t)$ is continuous at $t = t_0$ when $\vec{r}(t) \to \vec{r}(t_0)$ as $t \to t_0$.

NB: this is (once again) a straightforward extension of the definition of continuity given in Calculus I...

Because we proved the above properties of limits, we obtain the following properties of continuity.

Prop (BASIC PROPERTIES OF CONTINUITY): If scalar function $c : I \to \mathbb{R}$ and space curves $\vec{u}, \vec{v} : I \to \mathbb{R}^n$ are continuous at $t = a$, then so are the following.

1. $\vec{u}(t) + \vec{v}(t)$
2. $c(t)\,\vec{u}(t)$
3. $\vec{u}(t) \cdot \vec{v}(t)$
4. $\vec{u}(t) \times \vec{v}(t)$

NB: For each of the above, we know the corresponding limits exist. Continuity follows trivially in each case. ▨